Chapter 2

Sinusoidal steady state analysis

By

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AC Circuits Analysis

- The Node Voltage Method
- The Mesh Current Method
- Superposition of AC Sources
- Thevenin’s Theorems
- Norton’s Theorems

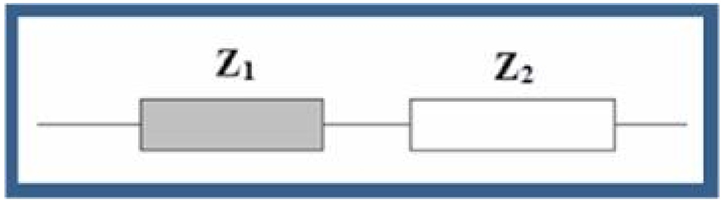
Nodal Analysis

Nodal analysis

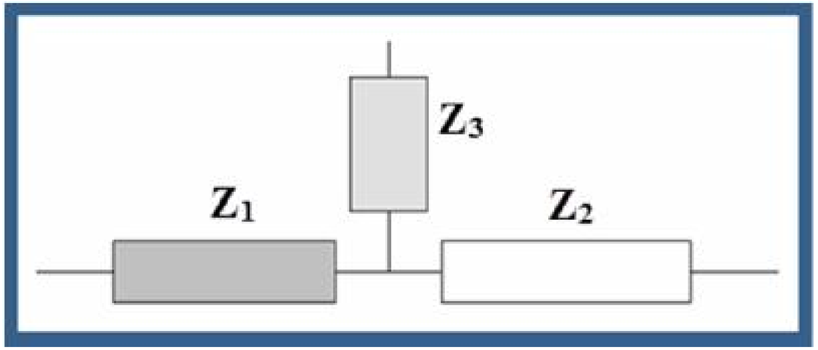
- The aim of nodal analysis is to determine the voltage at each node relative to the reference node (or ground).
- Once you have done this you can easily work out anything else you need.

Branches and Nodes

Branch: elements connected end-to-end, nothing coming off in between (in series)

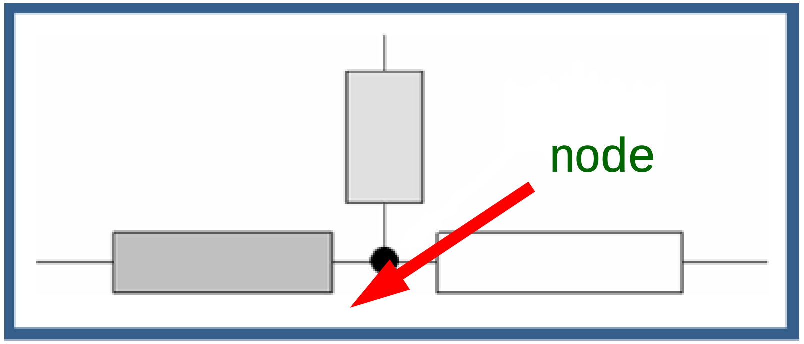


A single branch



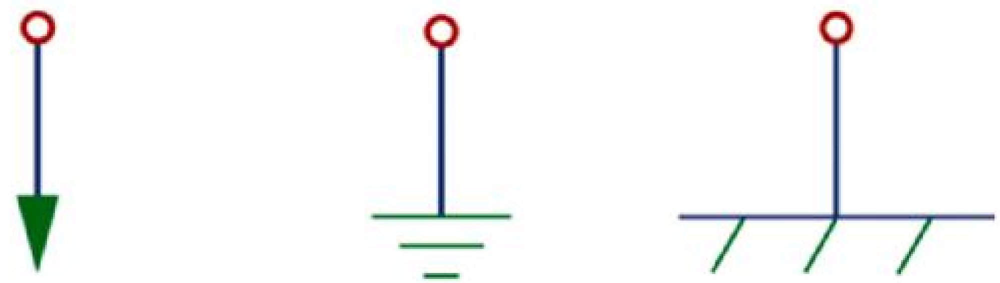
Not a single branch

Node: place where elements are joined—includes entire wire



Nodal analysis

Common symbols for indicating a reference node, (ground)

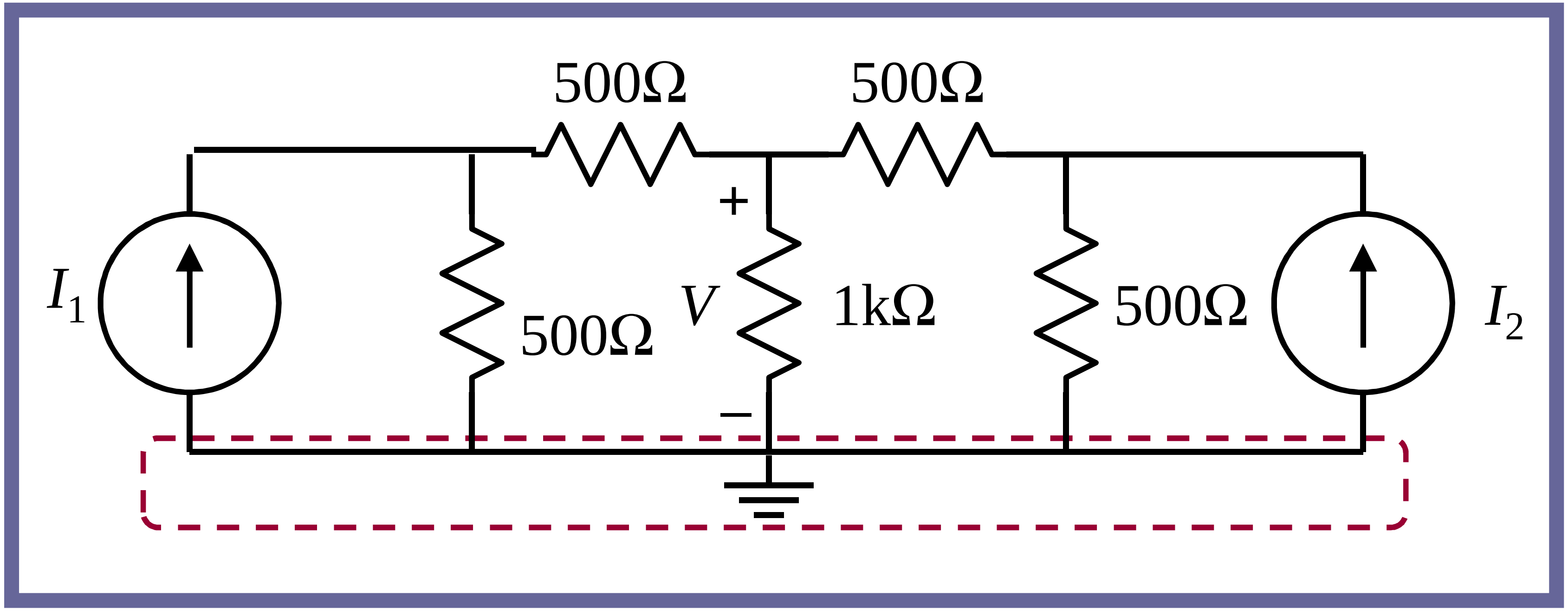


(a) (b) (c)

Steps of Nodal Analysis

1. Choose a reference (ground) node.
2. Assign node voltages to the other nodes.
3. Apply KCL to each node other than the reference node; express currents in terms of node voltages.
4. Solve the resulting system of linear equations for the nodal voltages.

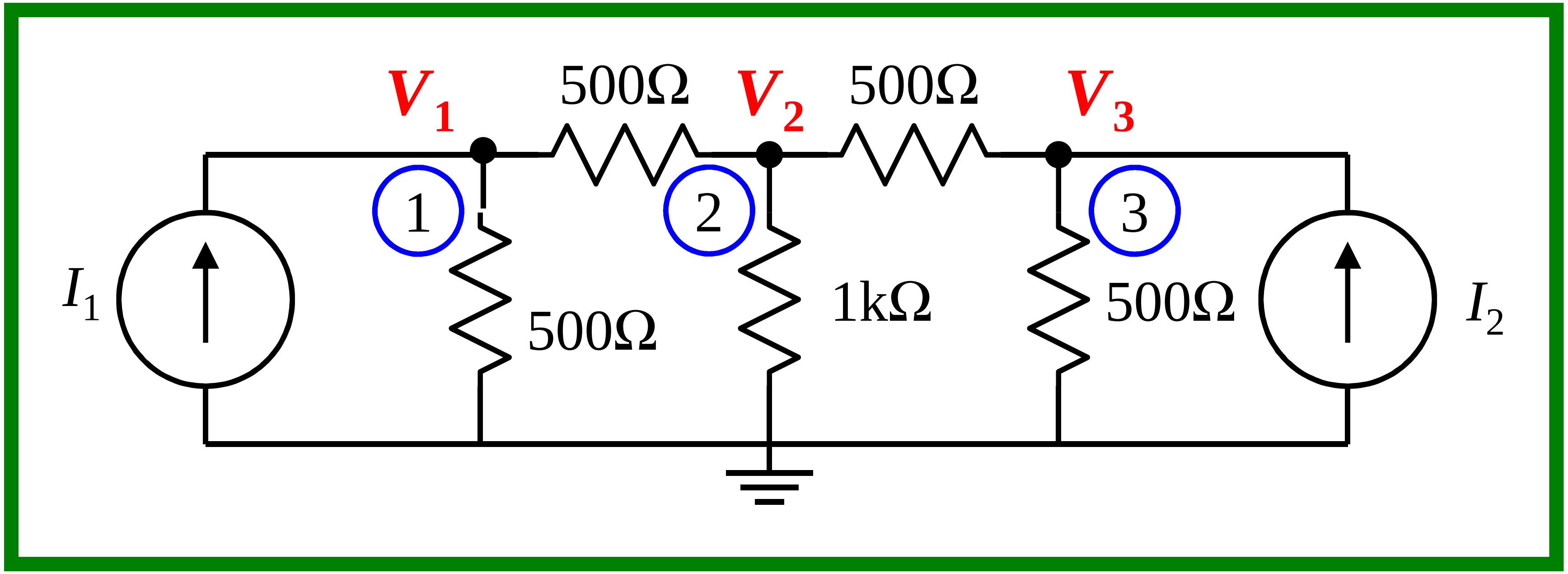
1. Reference Node



The reference node is called the *ground* node

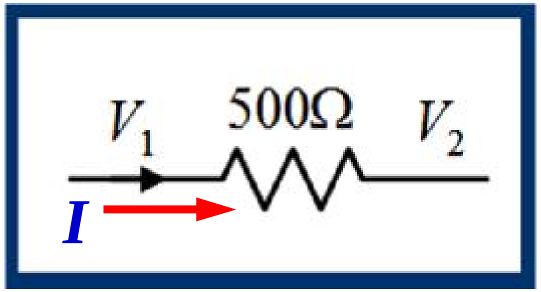
where $V = 0$

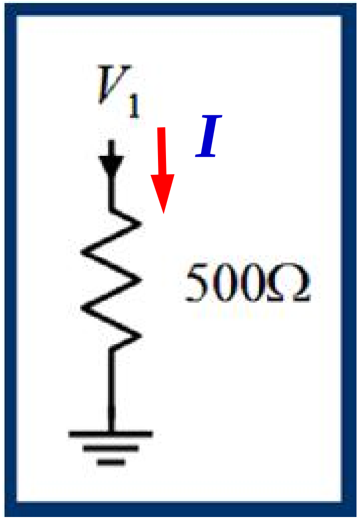
2. Node Voltages



V_1, V_2, and V_3 are unknowns for which we solve using KCL

2. Node Voltages

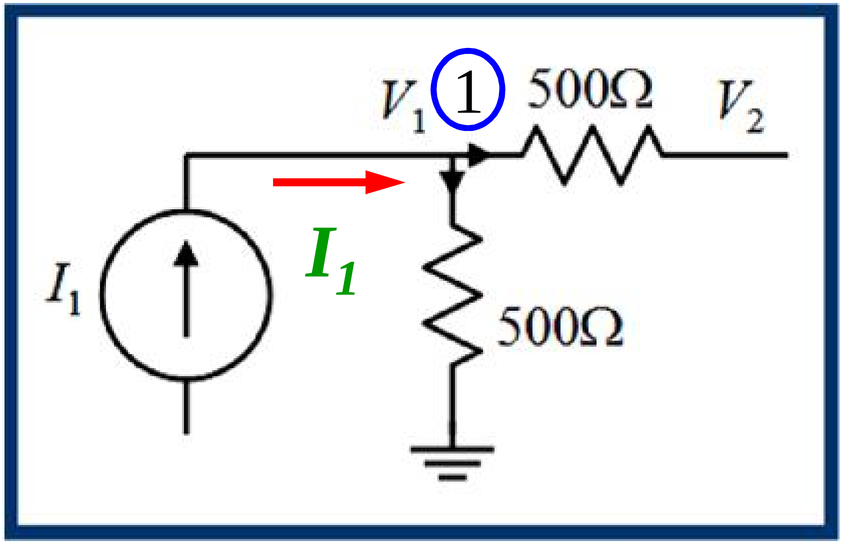




$$I = \frac{V_1 - V_2}{500\Omega}$$

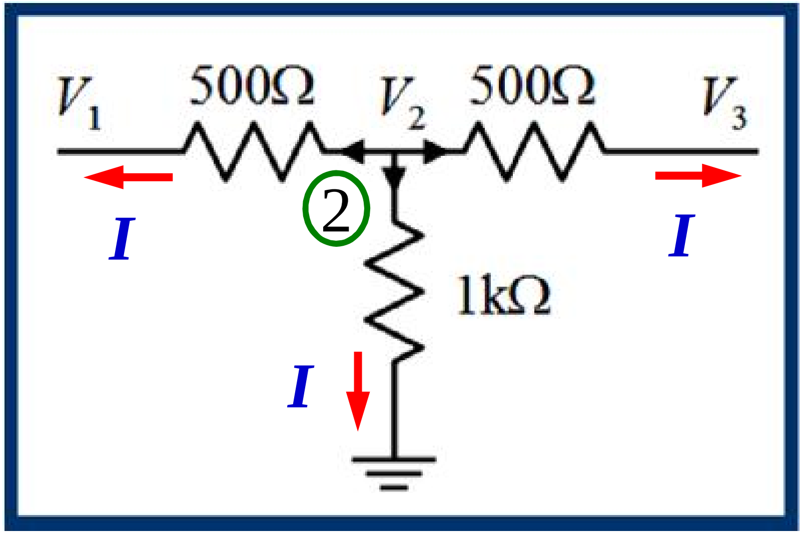
$$I = \frac{V_1}{500W}$$

3. KCL at Node 1



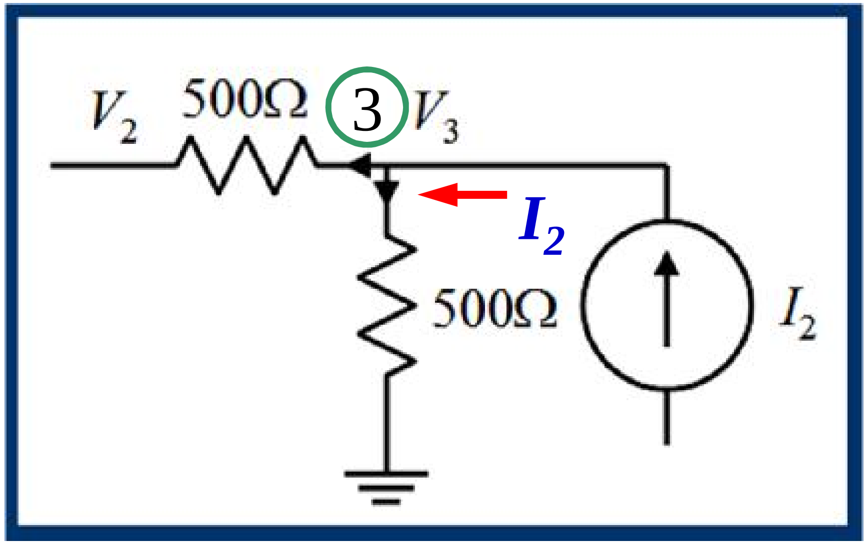
$$I_1 = \frac{V_1 - V_2}{500\Omega} + \frac{V_1}{500\Omega}$$

3. KCL at Node 2



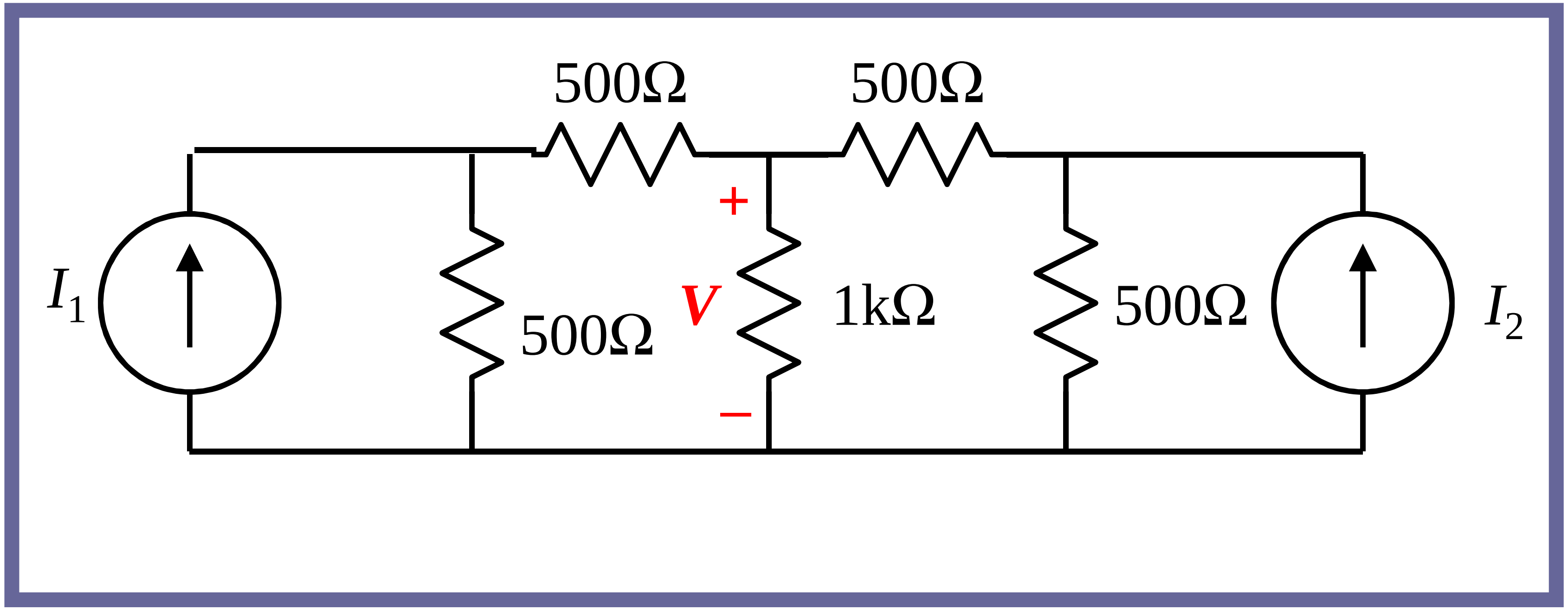
$$\frac{V_2 - V_1}{500\Omega} + \frac{V_2}{1k\Omega} + \frac{V_2 - V_3}{500\Omega} = 0$$

3. KCL at Node 3



$$\frac{V_3 - V_2}{500\Omega} + \frac{V_3}{500\Omega} = I_2$$

4. Summing Circuit Solution

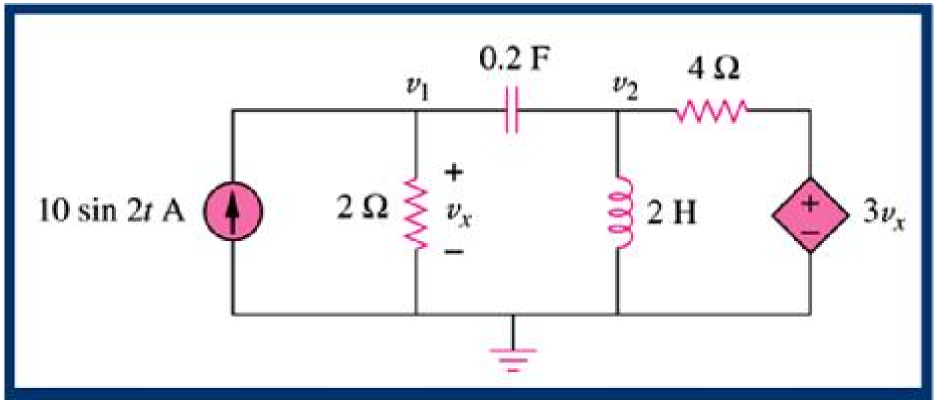


Solution: $V = 167I_1 + 167I_2$

Ex. 1: Find v_1 and v_2 using

nodal analysis

Solution

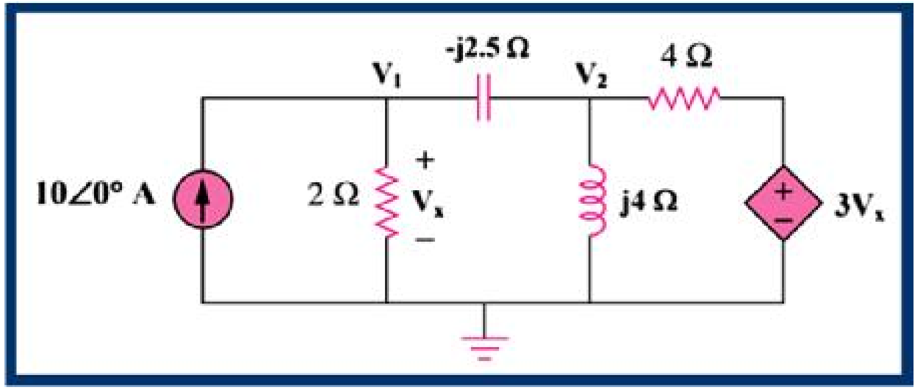


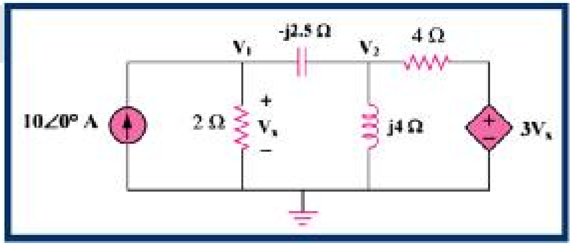
w = **2 rad/s**

L = 2 H → X_L **=wL= 4Ω**

C = 0.2F → X_C **=1/wC=2.5 Ω**

Hence, the circuit in the frequency domain is as shown below.





At node 1

$$10 = \frac{V_1}{2} + \frac{V_1 - V_2}{-j2.5}$$

$$100 = (5 + j4)V_1 - j4V_2 \quad \longrightarrow \quad \textbf{Eq. (1)}$$

At node 2

$$\frac{V_2}{j4} = \frac{V_1 - V_2}{-j2.5} + \frac{3V_x - V_2}{4} \qquad \text{where } V_x = V_1$$

$$-j2.5V_2 = j4(V_1 - V_2) + 2.5(3V_1 - V_2)$$

$$0 = -(7.5 + j4)V_1 + (2.5 + j1.5)V_2 \quad \longrightarrow \quad \textbf{Eq. (2)}$$

Put (1) and (2) in matrix form.

$$\Delta \leftarrow \begin{bmatrix} 5 + j4 & -j4 \\ -(7.5 + j4) & 2.5 + j1.5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

where $\Delta = (5 + j4)(2.5 + j.15) - (-j4)(-(7.5 + j4)) = 22.5 - j12.5$

$$\Delta = 25.74\angle -29.05°$$

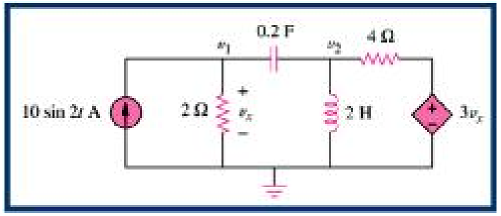
$$\Delta_1 = \begin{bmatrix} 100 & -j4 \\ 0 & 2.5 + j1.5 \end{bmatrix} = 250 + j150 = 291.55 \angle 30.96^\circ$$

$$\Delta = \begin{bmatrix} 5 + j4 & -j4 \\ -(7.5 + j4) & 2.5 + j1.5 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} 5 + j4 & 100 \\ -(7.5 + j4) & 0 \end{bmatrix} = 750 + j400 = 850 \angle 28.07^\circ$$

$$V_1 = \frac{\Delta_1}{\Delta} = \frac{291.55 \angle 30.96^\circ}{25.74 \angle -29.05^\circ} = 11.32 \angle 60.01^\circ \text{ V}$$

$$V_2 = \frac{\Delta_2}{\Delta} = \frac{850 \angle 28.07^\circ}{25.74 \angle -29.05^\circ} = 33.02 \angle 57.12^\circ \text{ V}$$



In the time domain,

$$v_1(t) = 11.32 \sin(2t + 60.01^\circ) \text{ V}$$

$$v_2(t) = 33.02 \sin(2t + 57.12^\circ) \text{ V}$$

Mesh Analysis

Mesh analysis

- Mesh analysis: another procedure for analyzing circuits, applicable to planar circuit.
- A Mesh is a loop which does not contain any other loops within it

Steps to Determine Mesh Currents

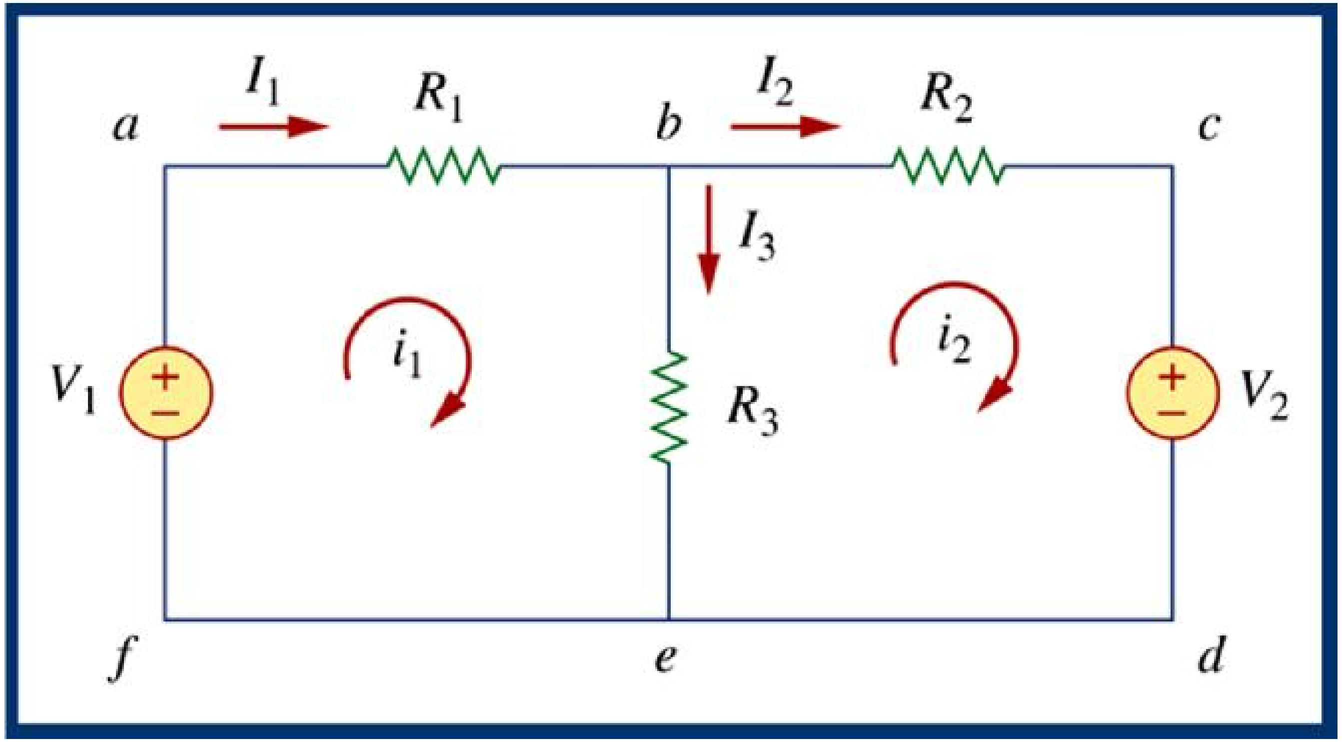
1- Assign mesh currents $i_1, i_2, .., i_n$ to the n meshes.

2- Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.

3- Solve the resulting *n* simultaneous equations to get the mesh currents.

1. Assign mesh currents

A circuit with two meshes.

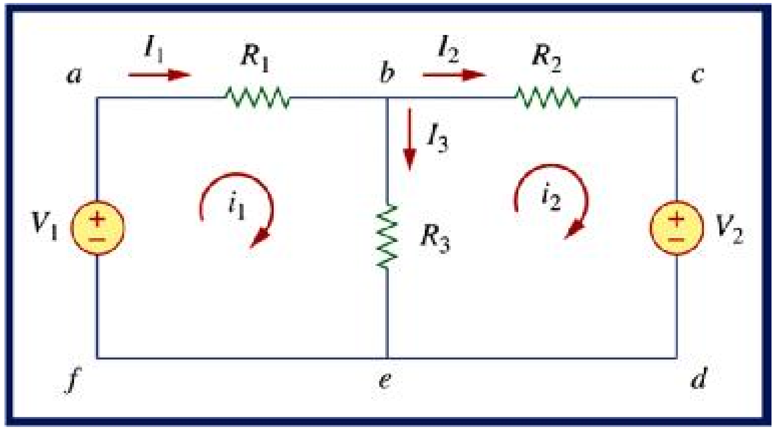


2. Apply KVL to each of the meshes

- For mesh 1, (abef)

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$$

$$(R_1 + R_3)i_1 - R_3 i_2 = V_1$$



- For mesh 2, (bcde)

$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$$

$$-R_3 i_1 + (R_2 + R_3)i_2 = -V_2$$

3. Solve the resulting equations

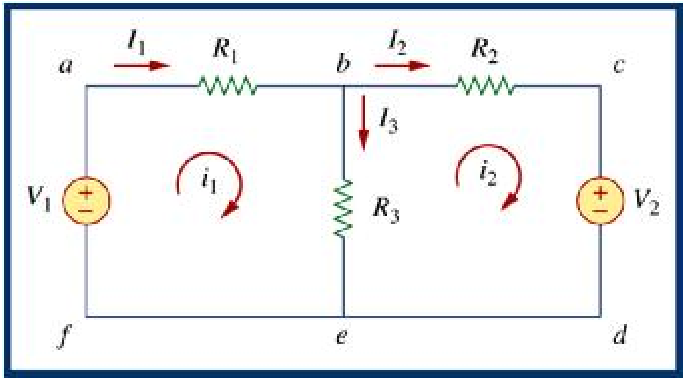
- Solve for the mesh currents in matrix form.

$$(R_1 + R_3)i_1 - R_3 i_2 = V_1$$
$$-R_3 i_1 + (R_2 + R_3)i_2 = -V_2$$

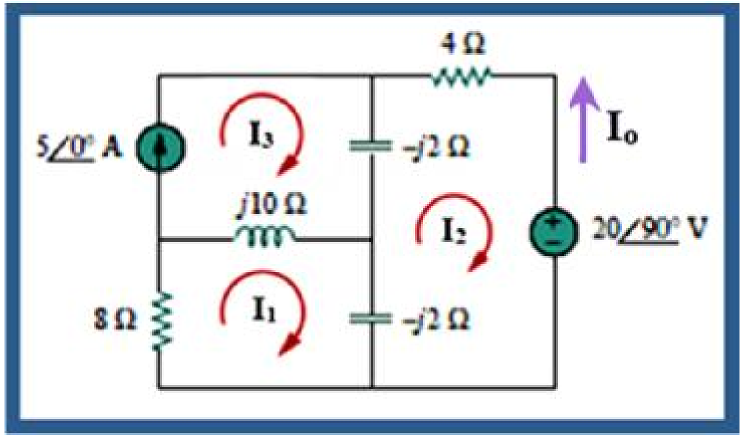
$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

- Use *i* for a mesh current and *I* for a branch current.

$$I_1 = i_1, \; I_2 = i_2,$$
$$I_3 = i_1 - i_2$$



Ex. 2: Determine I_o using Mesh analysis



Solution

Applying KVL to mesh 1

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0$$ → **Eq. (1)**

For mesh 2

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0$$ → **Eq. (2)**

For mesh 3, $\mathbf{I}_3 = 5$ → **Eq. (3)**

Substituting from Eq.(3) in Eq.(1) and Eq.(2)

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 = -j30$$

Putting mesh currents in matrix form.

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$
$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j30$$

$$\begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix} = (8+j8)\,(4-j4) - (j2 \times j2) = 68$$

$$\Delta_2 = \begin{bmatrix} 8+j8 & j50 \\ j2 & -j30 \end{bmatrix} = -j30\,(8+j8) - (j2 \times j50) = 416.17\underline{/-35.22^\circ}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\underline{/-35.22^\circ}}{68} = 6.12\underline{/-35.22^\circ}\ \text{A}$$

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\underline{/144.78^\circ}\ \text{A}$$

Superposition Theorem

Superposition Principle

The Superposition theorem, Enables us to find the response of the circuit to each source acting alone, and then add them up to find the response of the circuit to all sources acting together.

The Superposition principle**,** states that the voltage across (or the current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

Steps of Appling Superposition Analysis

1- Turn off all independent sources except one.

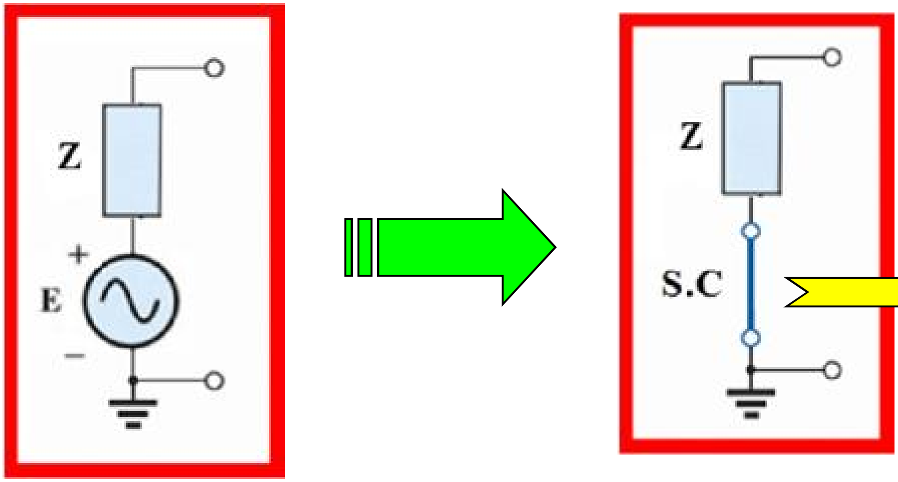
2- Find the output (voltage or current) due to the active source.

3- Repeat step 1and step 2 for each of the other sources.

4- Find the total output by adding algebraically all of the results found in steps 1, 2 & 3 above.

Turning sources off

Voltage source

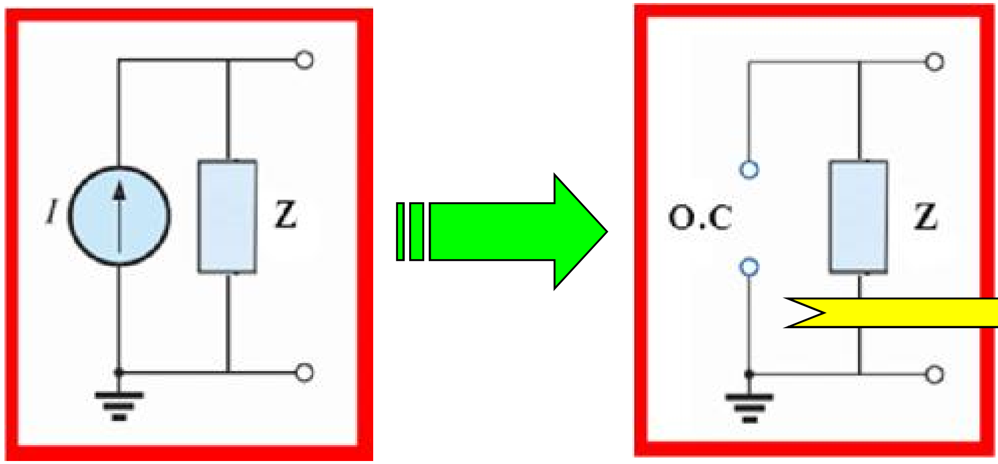


Turn off the voltage source (replace it by short circuits).

An short-circuit

$E = 0$

Current source

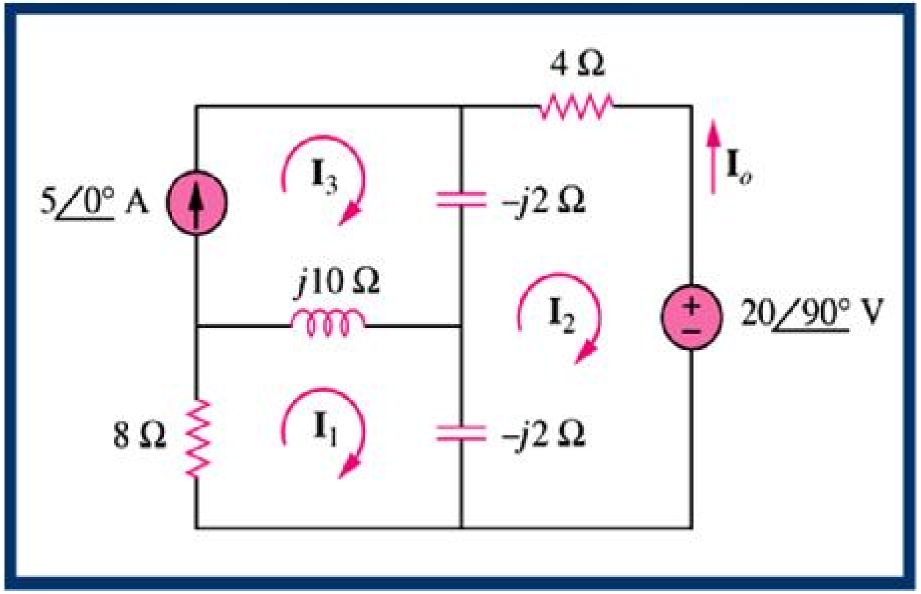


Turn off the current source (replace it by open circuits).

An open-circuit

$I = 0$

Ex. 3: Determine I_o using Superposition theorem.



Solution

First: Turn off the current source (replace it by open circuit) resulting Fig. (a) in order to calculate $\mathbf{I}_0'$

$$\mathbf{Z} = (-j2) \| (8 + j10) = 0.25 - j2.25$$

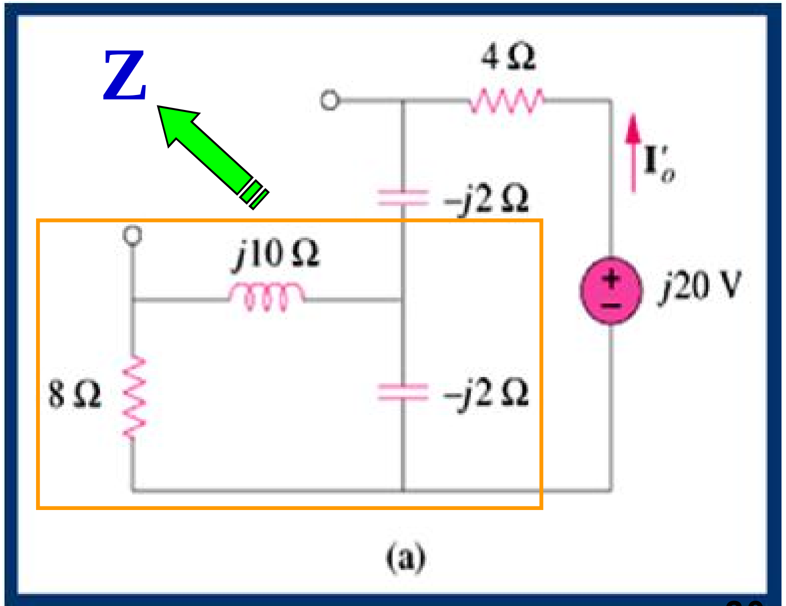
$$Z_T = (4 - j2) + Z$$

$$= (4 - j2) + (0.25 - j2.25)$$

$$= 4.25 - j4.25 \ \Omega$$

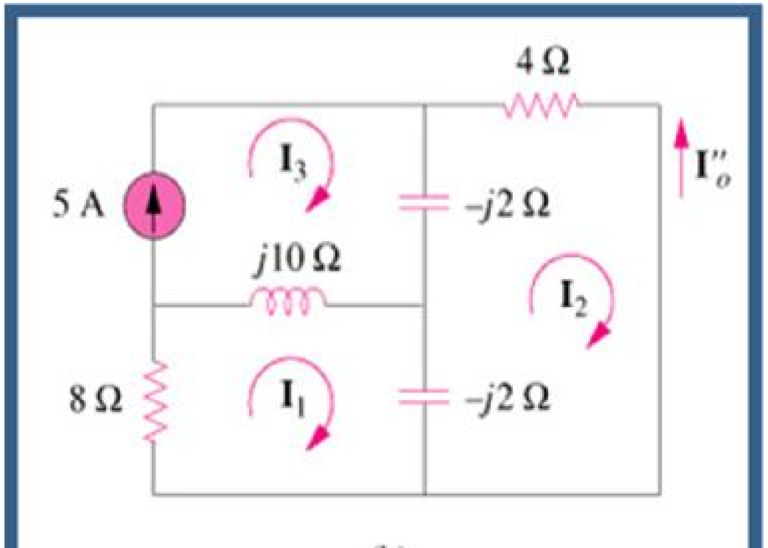
$$I_o' = \frac{j20}{Z_T} = \frac{j20}{4.25 - j4.25}$$

$$\mathbf{I}_0' = -2.353 + j2.353 \ \text{A}$$



(a)

Second: Turn off the voltage source (replace it by short circuits) resulting Fig. (b) in order to calculate $\mathbf{I}_0''$



(b)

To get $\mathbf{I}_0''$, Applying KVL to mesh 1

$$(8+j8)I_1 - j10I_3 + j2I_2 = 0$$

For mesh 2

$$(4-j4)I_2 + j2I_1 + j2I_3 = 0$$

For mesh 3

$$I_3 = 5$$

Putting mesh currents in matrix form.

$$\begin{bmatrix} 8+8j & j2 \\ j2 & 4-j4 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j10 \end{bmatrix}$$

$$\mathbf{I}_2 = 2.647 + j1.176\ \mathbf{A}$$

$$\mathbf{I}_0' = -2.353 + j2.353\ \mathbf{A}$$

$$\mathbf{I}_0'' = -\mathbf{I}_2 = -2.647 - j1.176$$

$$\mathbf{I}_0 = \mathbf{I}_0' + \mathbf{I}_0''$$

$$= (-2.353 - 2.647) + j(2.353 + 1.176) = -5 + j3.529$$

$$\mathbf{I}_0 = 6.12\angle 144.78^\circ\ \mathbf{A}$$

Thévenin's theorem

Thévenin’s theorem

Thévenin’s theorem, as stated for sinusoidal AC circuits, is changed only to include the term *impedance* instead of *resistance.*

Thévenin's theorem, for linear electrical networks states that any combination of voltage sources, current sources, and Impedances with two terminals is electrically equivalent to a single voltage source V_{th} in series with a single series Impedance Z_{th}.

Steps of Appling Thévenin's theorem

1- Identify and remove the load Impedance Z_L and Assign the load terminals by a-b

2- Look in the load terminals and calculate V_{Th} (V_{oc})

3- Remove all sources by replacing:

– voltage sources with a short circuit

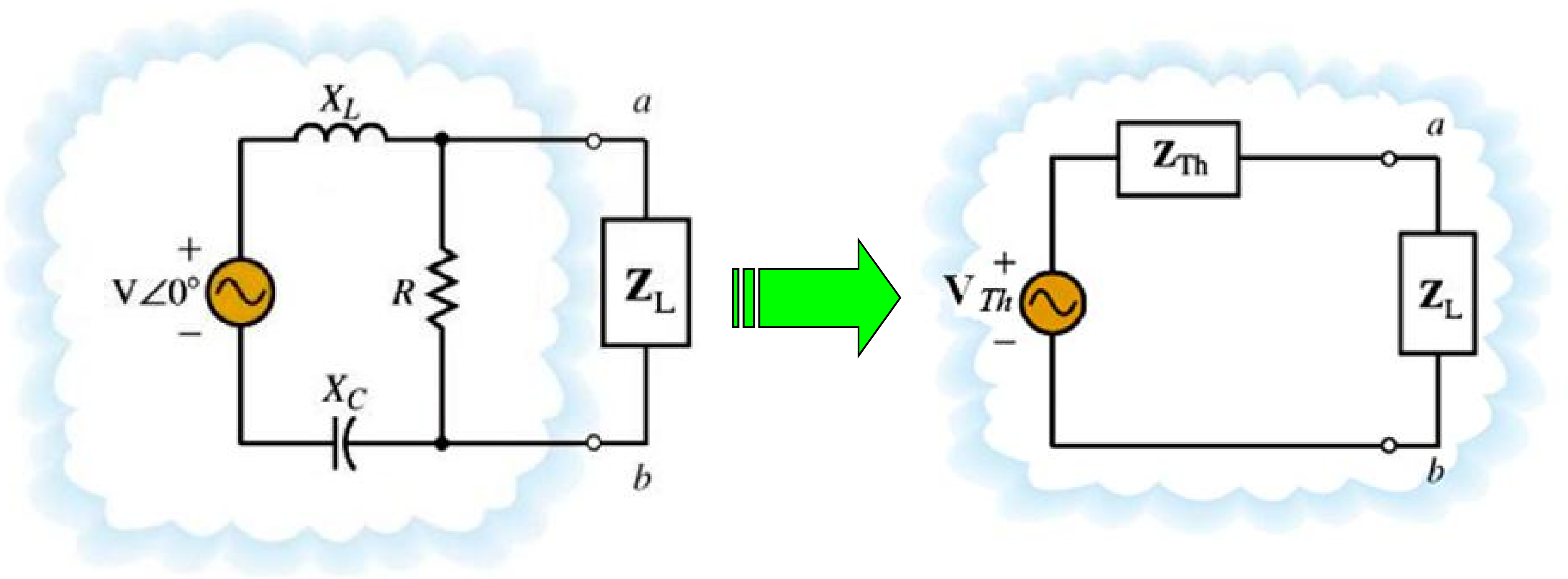
– current sources with an open circuit

4- Look in the load terminals and calculate Z_{Th}

5- Create a series circuit consisting of V_{Th} , Z_{Th} , and the load Z_L

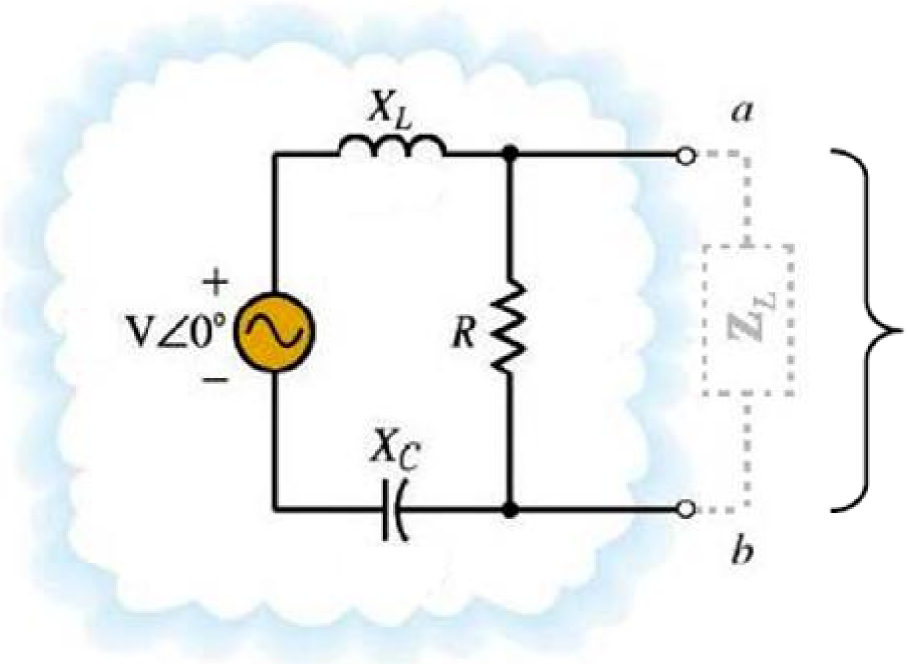
6- Calculate the load current I_L or voltage V_L as desired

Thévenin's theorem



- V_{Th} is the open circuit voltage at the terminals,
- Z_{Th} is the input or equivalent resistance at the terminals when the independent sources are turned off.

Determining E_{TH}

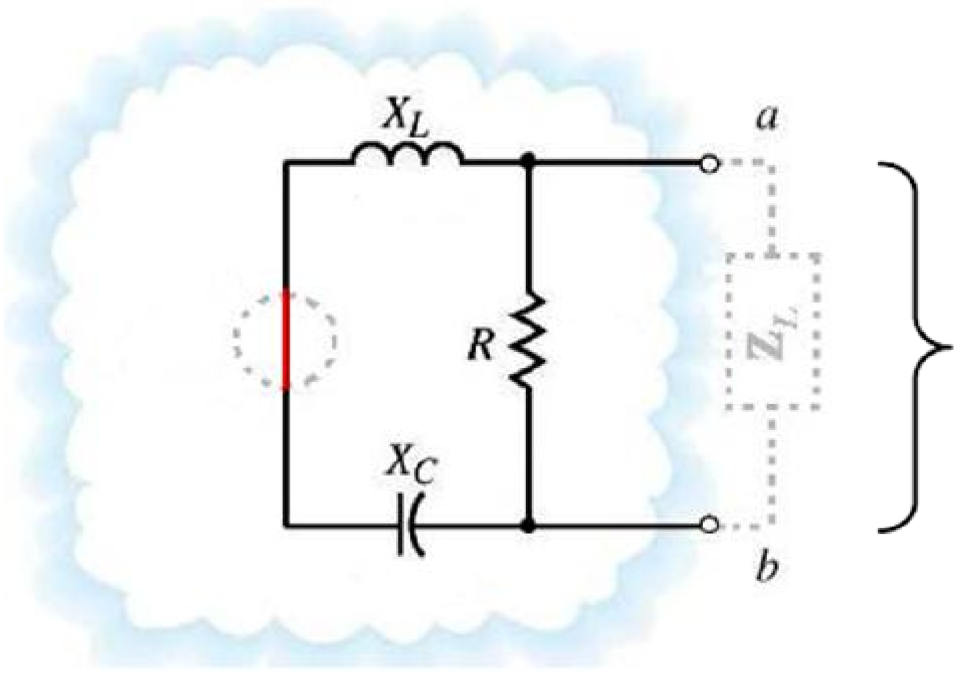


V_{th} **= open circuit voltage**

V_{Th} is the Open circuit voltage between the terminals a-b.

- Remove the load (open-circuit) and measure the resulting voltage.

Determining Z_{TH}



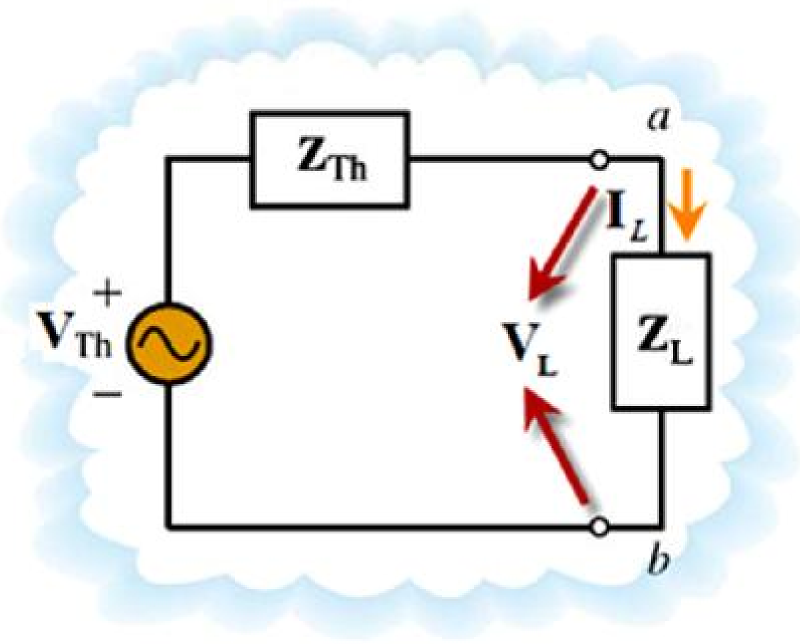
$\mathbf{Z_{th} = Z_{ab}}$

Z_{Th} is the impedance seen from the terminals when the sources are set to zero.

- With the load disconnected, turn off all independent sources.
- **Voltage sources** – 0 V is equivalent to a short-circuit.
- **Current sources** – 0 A is equivalent to an open-circuit.

Applying Thévenin equivalent

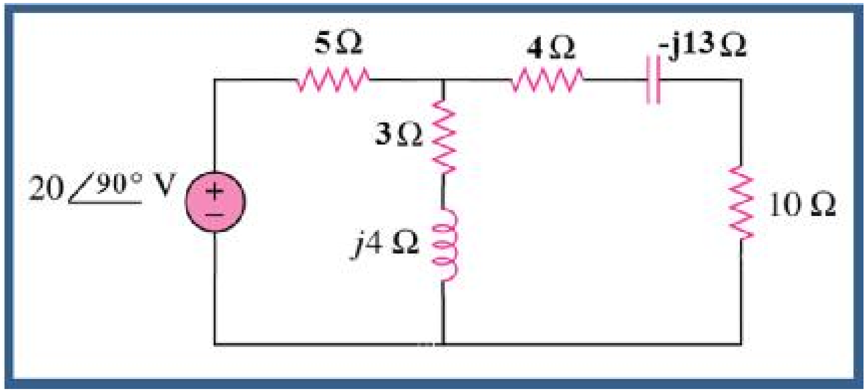
- Once $\mathbf{V}_{Th}$ and $\mathbf{Z}_{Th}$ have been found, the original circuit is replaced by its equivalent and solving for $\mathbf{I}_L$ and $\mathbf{V}_L$ becomes very simple.



$$\mathbf{V}_L = \frac{\mathbf{Z}_L}{\mathbf{Z}_{Th} + \mathbf{Z}_L}\mathbf{V}_{Th}$$

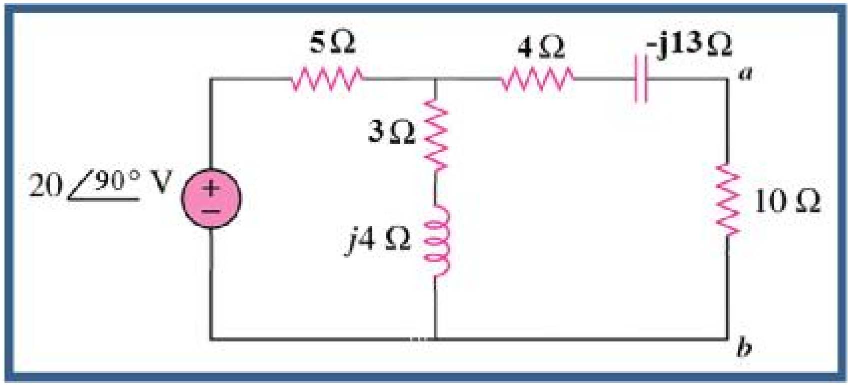
$$\mathbf{I}_L = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + \mathbf{Z}_L}$$

Ex. 5: Using Thevenin equivalent get the current in the 10 Ω resistor

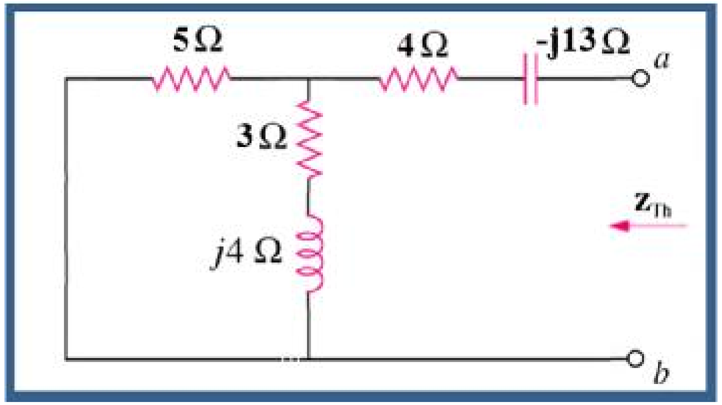


Solution

1- Assign the load terminals by a-b



2- Remove the load Impedance Z_L and Remove all sources



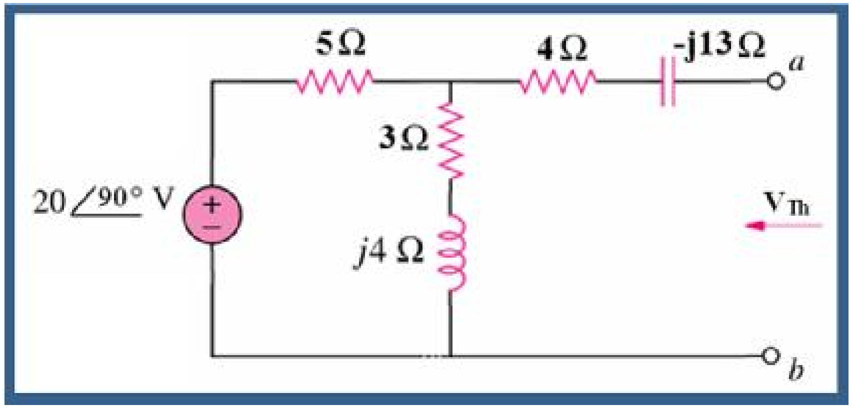
3- calculate Z_{Th}

$$Z_{Th} = \frac{5(3+j4)}{8+j4} + (4-j13) = 6.5 - j11.75\ \Omega$$

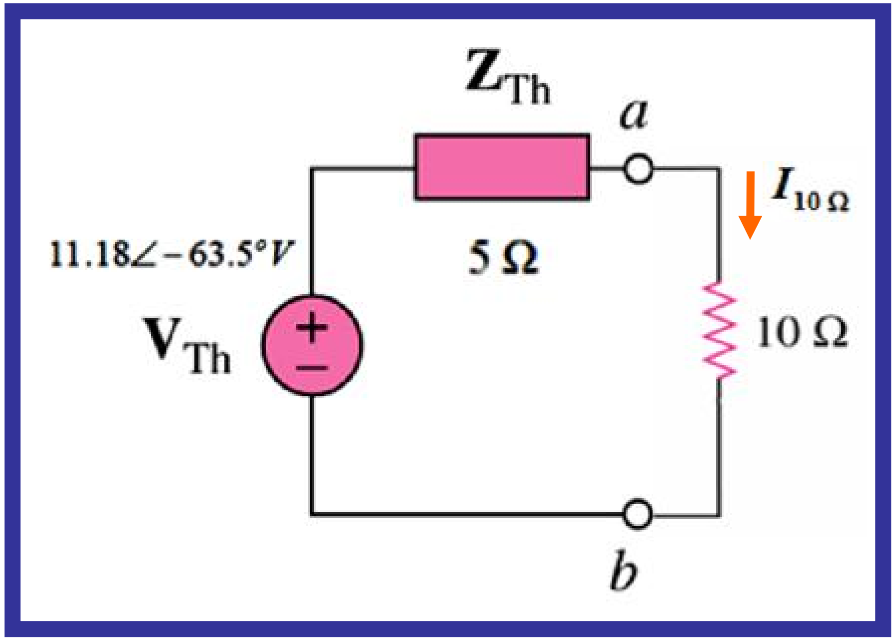
4- Open the load terminals and calculate

V_{Th} (V_{oc})

$$V_{Th} = V_{ab} = \frac{20\angle -90^o}{5+3+j4} x(3+j4) = 11.18\angle -63.5^o V$$



5- The equivalent Thevenin circuit as shown:



$$\mathbf{I}_L = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + \mathbf{Z}_L}$$

$$I_{10\,\Omega} = \frac{11.18\angle -63.5^o}{(6.5 - j11.75) + 10}$$

$$= \frac{11.18\angle -63.5^o}{20.256\angle -35.4^o}$$

$$I_{10\,\Omega} = \boxed{0.5519\angle -28.1^o\,A}$$

Norton theorem

Norton's theorem

Norton's theorem, as stated for sinusoidal AC circuits, is changed only to include the term *impedance* instead of *resistance.*

Norton's theorem, for linear electrical networks states that any combination of voltage sources, current sources, and Impedances with two terminals is electrically equivalent to a single current source I_N in parallel with a single series Impedance Z_N.

Steps of Appling Norton's theorem

1- Identify and remove the load Impedance Z_L and Assign the load terminals a-b

2- Short the load terminals and calculate I_N (I_{SC})

3- Remove all sources by replacing:

– voltage sources with a short circuit

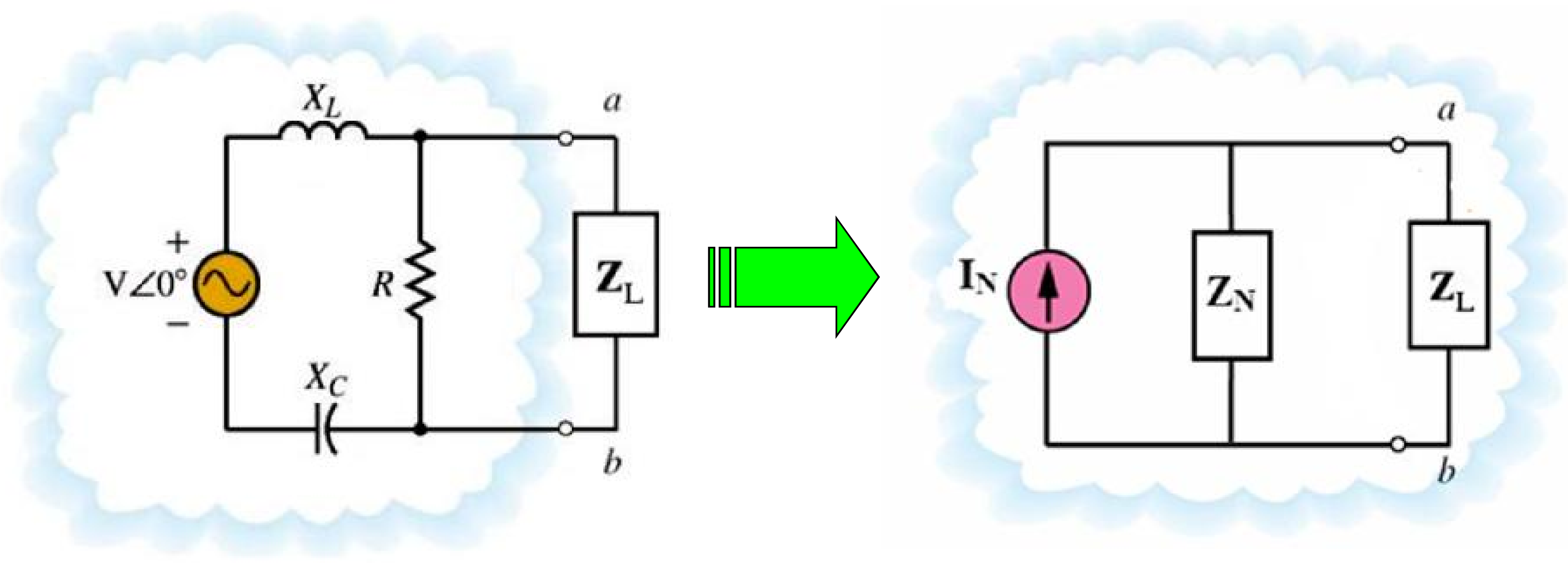
– current sources with an open circuit

4- Look in the load terminals and calculate Z_N

5- Create a parallel circuit consisting of I_N , Z_N , and the load Z_L

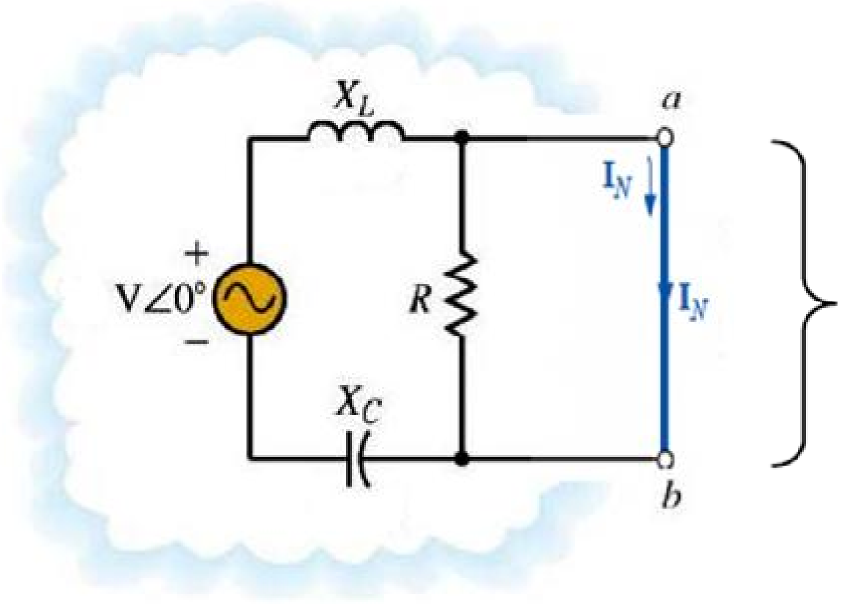
6- Calculate the load current I_L or voltage V_L as desired

Norton's theorem



- I_N is the short circuit current in the terminals a-b
- Z_N is the input or equivalent Impedance at the terminals a-b when the sources are turned off.

Determining I_N

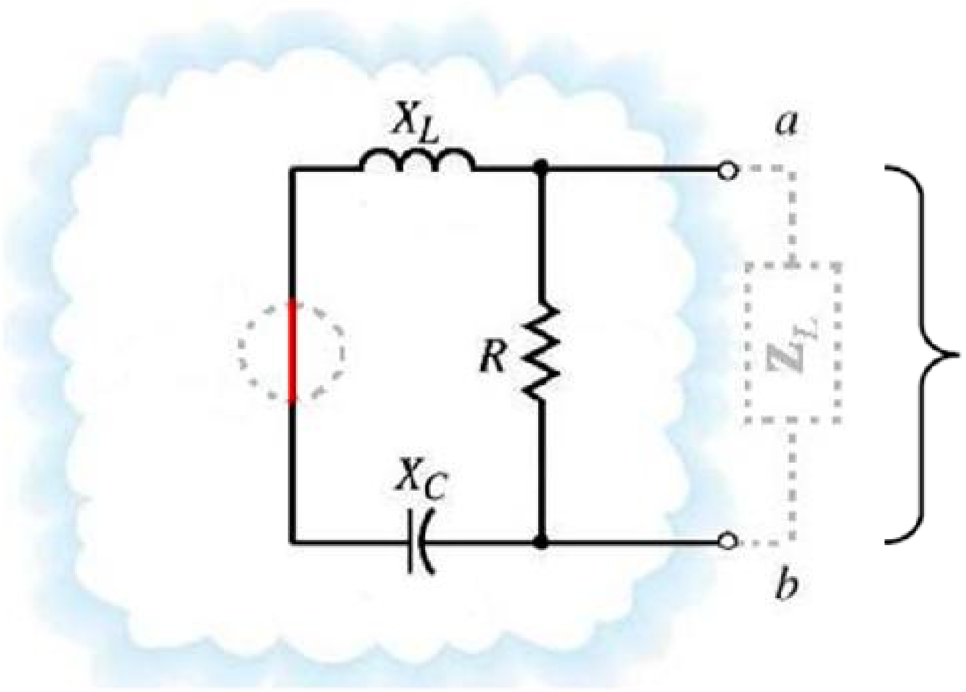


I_N = short circuit current

I_N is the Short circuit Current in the terminals a-b.

- Short the load terminals and measure the resulting current.

Determining Z_N



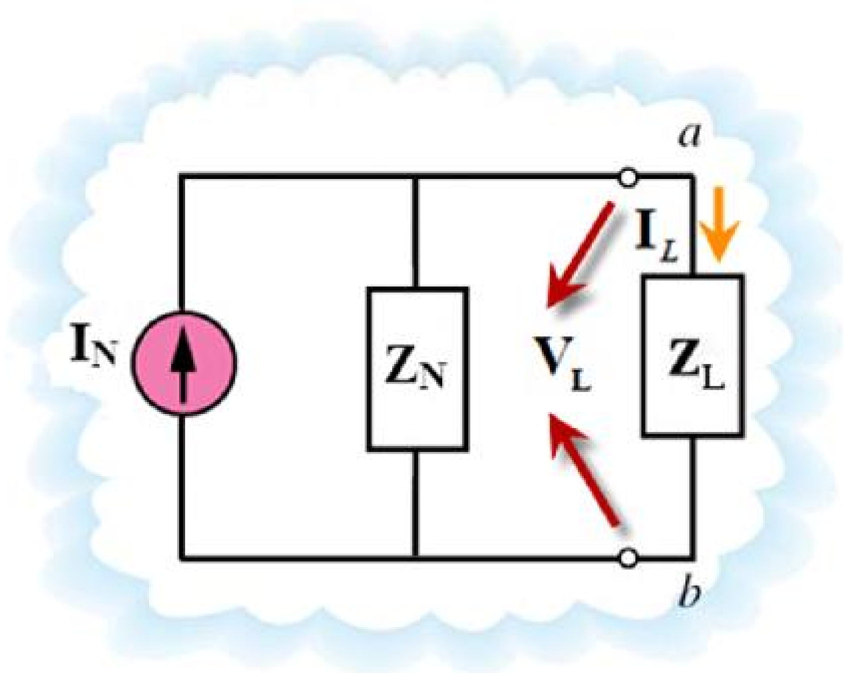
$\mathbf{Z_N = Z_{ab}}$

Z_N is the impedance seen from the terminals when the sources are set to zero.

- With the load disconnected, turn off all independent sources.
- **Voltage sources** – 0 V is equivalent to a short-circuit.
- **Current sources** – 0 A is equivalent to an open-circuit.

Applying Norton equivalent

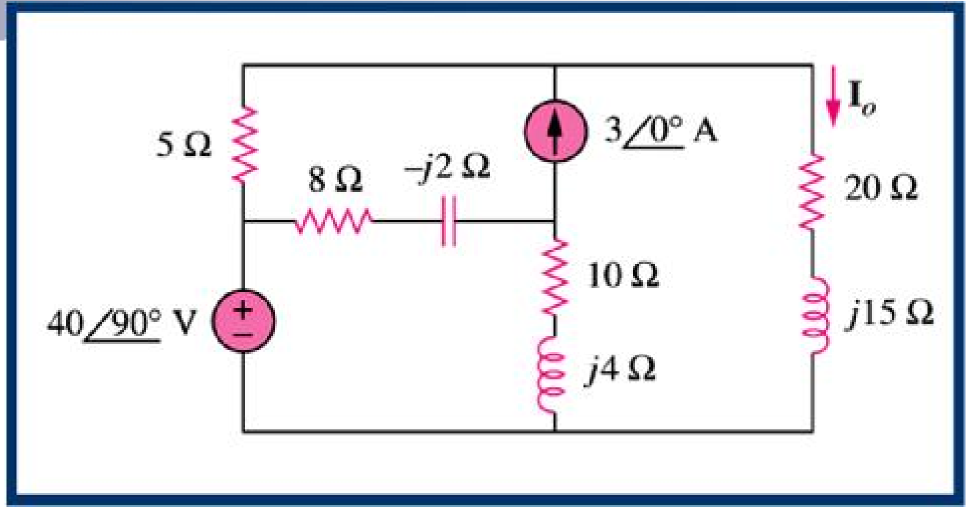
- Once $\mathbf{I_N}$ and $\mathbf{Z_N}$ have been found, the original circuit is replaced by its equivalent and solving for $\mathbf{I_L}$ and $\mathbf{V_L}$ becomes trivial.



$$I_L = \frac{Z_N}{Z_N + Z_L} I_N$$

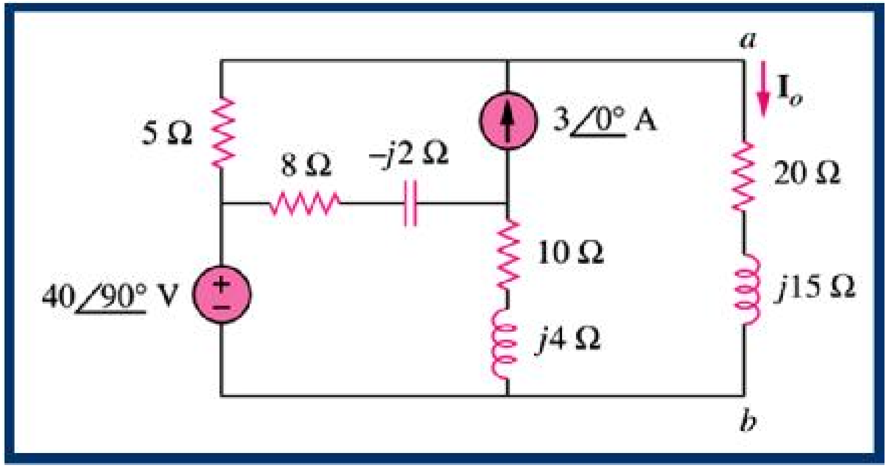
$$V_L = I_L\ Z_N$$

Ex. 5: Apply Norton theorem to get the current I_o

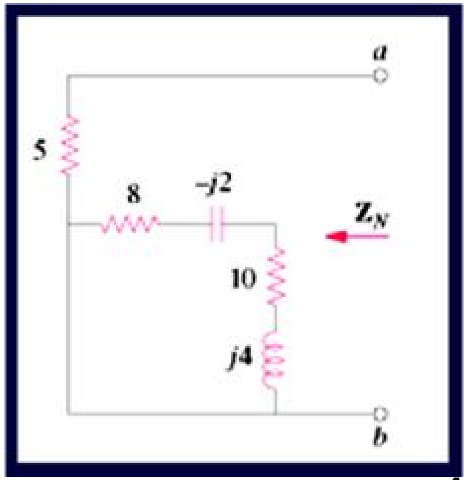


Solution

1- Assign the load terminals by a-b

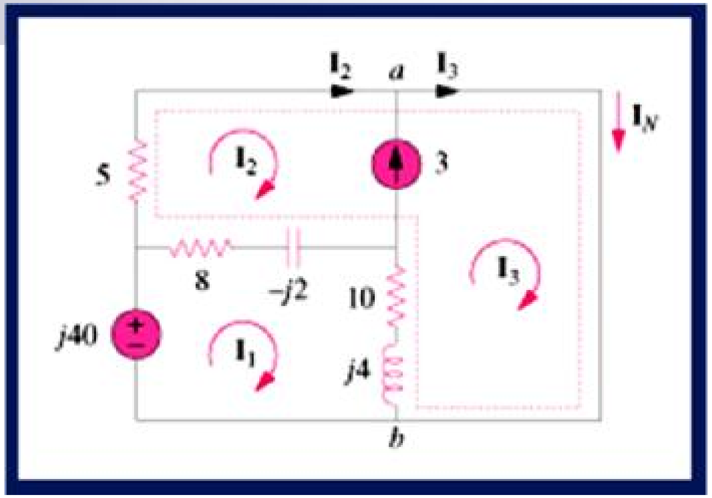


2- Remove the load Impedance Z_L and Remove all sources Ω



3- $\mathbf{Z}_N$ can be found easily, $\mathbf{Z}_N = 5\ \Omega$

4- Short the load terminals and calculate I_N (I_{SC}) using mesh analysis



Applying KVL for mesh 1

$$(18+j2)\mathbf{I}_1-(8-j2)\mathbf{I}_2-(10+j4)\mathbf{I}_3=j40$$

$$(18+j2)\mathbf{I}_1-(8-j2)\mathbf{I}_2-(10+j4)\mathbf{I}_3-j40=0$$ **Eq. (1)**

Applying KVL for mesh 2

$$(13-j2)\mathbf{I}_2-(10+j4)\mathbf{I}_3-(18+j2)\mathbf{I}_1=0$$ **Eq. (2)**

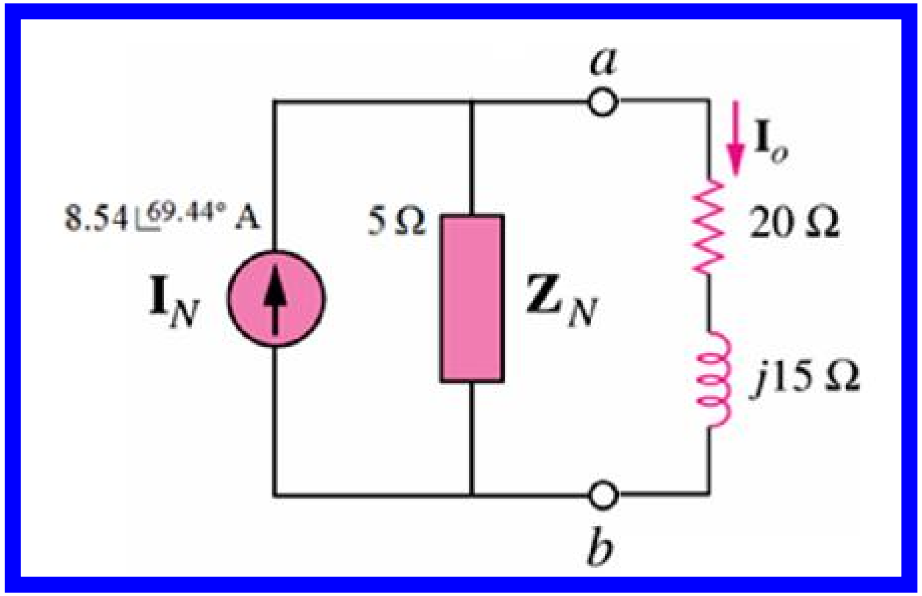
Applying KCL at node *a*

$$\mathbf{I}_3=\mathbf{I}_2+3$$ **Eq. (3)**

Substituting from Eq.3 in Eq.1 and Eq.2 and solve Eq.1and Eq.2 to get I_2 → $I_2 = j8$

$$\mathbf{I}_3=3+j8$$ → $I_N = I_3 = 3+j8 = 8.54\angle 69.44^\circ$ A

5- The equivalent Norton circuit as shown:



6- The required current $I_o = I_L$

$$I_L = \frac{Z_N}{Z_N + Z_L} I_N$$

$$I_o = \frac{5}{5 + 20 + j15} \times 8.54\angle 69.44° = 1.465\angle 38.48°\ A$$